FOURIER METHODS FOR COMPUTATIONAL ANALYSIS OF ENHARMONICISM AND OTHER HARMONIC PROPERTIES

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RÉSUMÉ

Diverses recherches antérieures ont démontré qu'on peut caractériser le vif développement du langage harmonique à la fin du XVIII[e] siècle par une diminution quantitative d'une grandeur appelée *diatonicité*. Celle-ci est définie en appliquant la transformée de Fourier discrète à une distribution des classes des hauteurs, c'est-à-dire un vecteur qui assigne à chaque classe de note une valeur liée à sa fréquence d'occurrence dans une passage musical. Cette technique de transformée de Fourier permet de dériver de nombreuses qualités harmoniques. Le cinquième coefficient est particulièrement important dans un contexte tonal, car il représente la diatonicité.

Cette recherche utilise la transformée de Fourier discrète pour analyser les évolutions du langage harmonique dans les quatuors à corde par Haydn, Mozart et Beethoven de 1780 à 1810. Elle confirme la spécificité de la qualité diatonique, qui permet de suivre les changements de style harmonique au cours de cette période et de différencier les sections de la forme sonate, mais on découvre aussi que la qualité *octatonique* joue un rôle de plus en plus important au fur et à mesure de cette période.

1. INTRODUCTION

The late eighteenth and early nineteenth century are generally acknowledged to be a period of rapid development of harmonic language in the classical style. Over this period, composers use a wider range of keys and enharmonic techniques become increasingly important. In this paper I propose a method of measuring enharmonicism and other types of harmonic variability using on the Fourier transform on pitch-class vectors, and apply it to eleven string quartet movements by Haydn, Beethoven, and Mozart, over the three decades from 1780 to 1810.

The term enharmonicism is most often associated with techniques where a respelling of a note or chord is essential to show multiple harmonic functions it represents in a particular musical passage, for instance the respelling of a diminished seventh to function in two different keys. Another distinct type of enharmonicism occurs when a sequence of chords or keys tours the circle of fifths, so that a respelling is necessary somewhere in the sequence to return to the original key in its original spelling. Because spelling is equivalent to orientation around the circle of fifths (sharpward or flatward), both types of enharmonicism can be generalized as spread on the circle of fifths, which is operationalized here using the fifth Fourier coefficient. We find, in fact, in the examples below, that methods and uses of enharmonicism do not always fit neatly into these two categories, but all of them are captured by the generalization using the fifth Fourier coefficient.

In addition to providing a convenient generalization of enharmonicism, the discrete Fourier transform also isolates a number of harmonic qualities of potential interest, some of which, such as the third coefficient, have theoretical significance for tonality. The following study analyzes the Fourier coefficients of a windowed analysis of string quartet first movements, and how the range of values interacts with formal section, composer, and date of composition. In addition to the fifth coefficient, the analysis finds significant trends in the fourth coefficient, or octatonic quality.

2. METHOD

2.1. Fourier analysis of pitch-class vectors

The main mathematical tool for this research is the *discrete Fourier transform on pitch-class vectors*. A *pitch-class vector* is a twelve-place vector, where the first entry gives a weighting to the pitch-class C, the second to C♯, and so forth. A pitch-class set can be represented as a pitch-class vector by assigning ones to each pitch class present and zeros otherwise (the *characteristic function*). While pitch-class vectors have a variety of applications, they are used here to represent the frequency of occurrence of a pitch class in a musical passage. Applying the discrete Fourier transform to a pitch-class vector reveals the presence of periodic components.

For pitch-class vector $A = (a_0, a_1, a_2, \ldots, a_{11})$, the Fourier transform $\hat{A} = (\hat{a}_0, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{11})$ of A is given by ($\forall k : 0 \leq k \leq 11$),

$$\hat{a}_k = \sum_{j=0}^{11} a_j e^{-i2\pi kj/12} \quad (1)$$

$$= \sum_{j=0}^{11} a_j \left(\cos(2\pi kj/12) + i \sin(2\pi kj/12)\right) \quad (2)$$

The use of this method for harmonic analysis is discussed at length by Amiot, Quinn, Yust, and others [2, 6, 16, 19, 20, 21], in music cognition research by Krumhansl and Cuddy et al. [13, 9], and for algorithmic generation of chord progressions by Bernardes et al. [5]. Each $\hat{a}_k$ is a complex number, where the distance from the origin, the *magnitude* $|\hat{a}_k|$, gives the strength of that periodic component, and the angle $\arg(\hat{a}_k)$ gives its phase. Of particular interest are $\hat{a}_5$ and $\hat{a}_3$. The fifth coefficient, $\hat{a}_5$, gives the *diatonicity* of the pitch-class vector (see [3]). Its magnitude is the strength of diatonicity, and its phase indicates the nearest diatonic collection, or the best key signature for it. The third coefficient, $\hat{a}_3$, isolates triadic relationships, and distinguishes chords in a key by harmonic function. These two components consistently account for the majority of the power in pitch-class counts of whole pieces, or long passages, of tonal music. [22, 23]

Because the pitch-class vector A is real-valued, coefficient $\hat{a}_{12-k}$ is equal in magnitude and opposite in phase to $\hat{a}_k$. Therefore, coefficients 7–11 can be ignored.

2.2. Windowing and pitch-class counting procedure

The present study uses a windowing procedure to derive a series of pitch-class vectors from a musical score. The windows vary from 2 to 32 quarter notes in length, and are taken every quarter note. We begin with a list of *pitches* (N.B. not pitch *classes*) occurring within each quarter-note beat. Note repetitions and durations within the beat are ignored, but doublings in multiple octaves are retained. Then this is reduced to a pitch-class vector that counts the number of occurrences of each pitch-class (i.e. the number of octaves it occurs in). This gives a good measure of the harmonic importance of each pitch-class (see [1, 11]), while eliminating possible over-counting of pitches due to rearticulation (e.g., a trill) or undercounting of pitches due to, e.g., written out staccato. The pitch-class vectors for each beat within the window are then summed over the whole window.

After taking the DFT, each coefficient is normalized by total *power*, which is the sum of the squared magnitudes of all the components ($\sum |\hat{a}_k|^2$). By one of the fundamental theorems of Fourier analysis (Parseval's) this is equivalent to 12 times the sum of the squared magnitudes of the pitch-class weights of A ($12 \sum a_k^2$). Normalization neutralizes the effect of large numbers of pitches in a passage (due, e.g., to an ornate melody), or especially heavy emphasis on one or a few pitch-classes, both of which otherwise could lead some data points to have much higher magnitudes simply for textural rather than harmonic reasons.

2.3. Data set

The data set comes from the first movements of eleven string quartets by Haydn, Mozart, and Beethoven, composed between 1781 and 1809. The quartets chosen are all of those with sonata form first movements in 4/4 or 2/4 time available in the Yale Classical Archives corpus (ycac.yale.edu). The time signature constraint is imposed so that the match of window sizes to measure lengths is consistent. I used five window sizes: 2, 4, 8, 16, and 32 quarter-notes.

Each piece was divided into three parts–exposition, development, and recapitulation–before performing the data analysis. In all but one case (Beethoven Op. 59 no. 1) the end of the exposition is indicated in the score by a repeat sign. The recapitulation begins from the return of the main theme material in the home key (unambiguous in all cases) and runs through the end of the piece. Any codas are therefore grouped with the recapitulation. Where alternate endings exist (either at the end of the exposition or coda) the second ending was used.

2.4. Analysis of convex hulls: Area and distance from zero

For each DFT coefficient, the data consists of a point in complex space for each window. To find the overall range of activity for each formal section of each piece, I collected the area and centroid of the convex hull for all of the points in that section and whether the origin falls within that convex hull. The area of the convex hull and its distance from the origin together roughly determine the range of phase values that occur in the passage. The range of phase values increases as the area becomes larger, and as the distance from the origin gets smaller. Figures 1 and 2 illustrate this with the $\hat{a}_5$ values for the exposition and development sections of Haydn's Op. 50/1 string quartet's first movement. Each sixteen quarter-note window, incremented by a quarter note, is a data point (lines and numbering in the figure show the temporal evolution), and the extreme data points determine the convex hull. The range of activity in the exposition covers about three key signatures of the total phase range of the space, whereas the development almost includes the origin, covering not quite half of the full enharmonic cycle (from D♭ major to C major / D minor). The difference between them is primarily that the range of activity in the development is closer to the origin.

Previous studies have averaged magnitudes (distance from zero) and phase values of Fourier coefficients over pieces or musical passages ([22, 23]). These quantities are closely related to the distance of the convex hull centroid from zero and the area of the convex hull respectively. The difference of looking at the convex hull is that it prioritizes the extrema of the passage, which may reflect moments of particular musical significance. If these extrema are isolated events, then this can be detected by the effect of increasing window size, which averages nearby points and will have the effect of neutralizing more isolated extreme values.

Four coefficients are considered here, $\hat{a}_2$, $\hat{a}_3$, $\hat{a}_4$, and $\hat{a}_5$. The fifth coefficient, as a measure of diatonicity, is of particular interest. The third has also been shown to be important to tonality, relating to harmonic function. The second also tends to be large in tonal distributions, but is dependent on the third and fifth (and hence possibly a sort

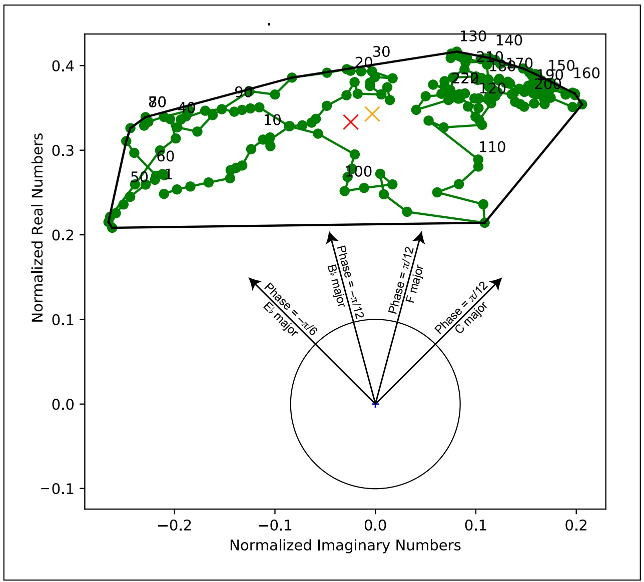

Figure 1. The exposition of Haydn's Op. 50/1 (1st mvt.) in $\hat{a}_5$-space, analyzed with a 16-beat window. Each ten quarter-notes are numbered, and successive windows are connected by a line. The convex hull is shown, as well as the average value (orange X) and centroid (red X).

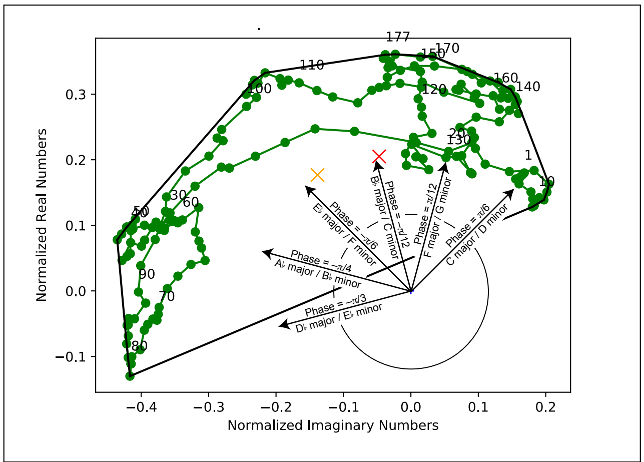

Figure 2. Development of Haydn's Op. 50/1 (1st mvt.) in $\hat{a}_5$-space, analyzed with a 16-beat window.

of mathematical artifact). [22, 23] The fourth coefficient is also large for triads and seventh chords, but tends to disappear in multi-measure tonal distributions. Theoretically it could play some role in harmonic function, although if so, this has not yet been demonstrated empirically.

3. RESULTS

The first question of interest is how the window size affects these values. The trends were consistent across formal sections, so they are averaged over them here. As the window gets larger, the area of the convex hull will necessarily decrease, because the points will get closer together. However, the area can decrease at different rates depending on how rapidly the values for that coefficient fluctuate over time. If they tend to move large distances over short periods of time, increasing window size will

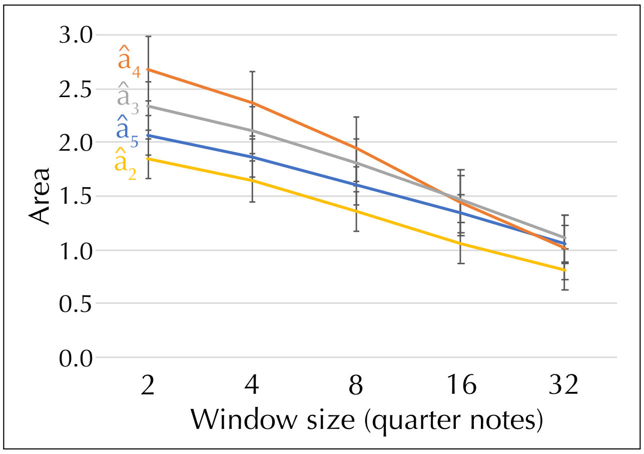

Figure 3. Change in area across coefficients for different window sizes, with standard error

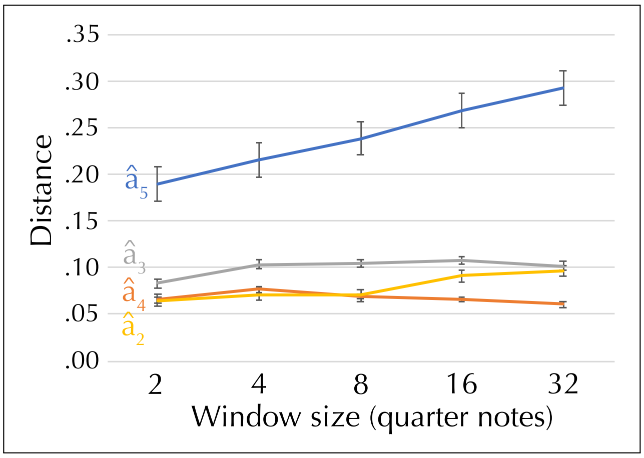

Figure 4. Change in distance from the origin across coefficients for different window sizes, with standard error

have a larger effect on the area. Figure 3 shows a tendency for $\hat{a}_4$ to decrease area more rapidly with change of window size than the other coefficients. Also, we see that the range of activity is lowest for $\hat{a}_2$ and $\hat{a}_5$ (large window sizes in development sections are an exception where $\hat{a}_5$ has a similar range of activity to $\hat{a}_3$ and $\hat{a}_4$). The range of activity is higher for $\hat{a}_3$ and $\hat{a}_4$.

The data for the distance of the centroid from the origin, shown in figure 4 (also consistent across formal sections), clearly shows that $\hat{a}_5$, diatonicity, is special. Not only is it consistently further from the origin, something shown by previous studies [22, 23], but it also shows a clear strong trend towards increasing distance from the origin with larger window size. The greater distance of $\hat{a}_3$ (and $\hat{a}_2$ at larger window sizes) compared to $\hat{a}_4$ is also consistent with previous findings. What previous studies do not show, because they deal mostly with larger window sizes (in particular entire pieces) is that $\hat{a}_2$ only gets larger as the window size increases to four or more measures.

The next question of interest is whether we can detect stylistic changes over this 30-year period. A previous study [23] found a very clear trend of decreasing diatonicity ($|\hat{a}_5|$) in a large corpus for music from the sixteenth through the nineteenth century. That analysis was based on pitch-class distributions taken over entire pieces, and also a single large window from the beginning and end of each piece. The analysis here considers a windowing of each formal section, and a range of $\hat{a}_5$ values (represented by the convex hull).

To evaluate this question, I ran a series of multiple regressions on the area and distance data for a single moderate window size of 8 quarter notes, with factors of (i) coefficient number (2–5), (ii) date of composition, (iii) composer, (iv) formal section, (v) mode, and (vi) length of the passage in quarter notes [1] , and second and third order interactions between these, and used a backward elimination method to find an efficient model [2]. The resulting models are given in tables 1-3.

The distance model in table 1 includes factors of coefficient number, date, formal section, and mode, and interactions of all of these with coefficient number (adjusted $R^2 = .827$, $p < .0001$). Significant βs appear for all the interactions and for the simple effect of coefficient number. Almost all of these involve $\hat{a}_5$, which is a) larger (farther from 0) overall, b) decreases with date of composition, and c) is smaller for developments and minor keys. Since distance of the centroid from the origin is closely related to the magnitude of a coefficient for the pitch-class distribution of the whole piece, these results are largely consistent with results from [23], which found overall consistently large diatonicity ($|\hat{a}_5|$) values compared to other coefficients, decreases of diatonicity over the eighteenth and nineteenth centuries, and greater diatonicity in major than minor. The finding of lower diatonicity in development sections is new, but unsurprising. The other results from the distance data involve $\hat{a}_4$, which is lower overall, and, interestingly and unexpectedly, gets larger over the 1780–1810 period, as $\hat{a}_5$ gets smaller.

The model in table 1 is not actually the one that resulted from the backward elimination procedure, which instead gave a model with length and length×coefficient rather than date and date×coefficient. However, because length and date are highly correlated, and date provides a better explanation for the observed effects (it is not obvious why length alone would influence the distances), I checked the model in table 1 and got almost identical results, with a slight improvement in multiple R^2 and adjusted R^2. Table 2 gives coefficients for the alternate model with length and length×coefficient (the coefficients not involving length or date do not meaningfully change). The only difference in the model with length is that only the interaction with $\hat{a}_5$ is significant ($\hat{a}_5$ gets smaller for longer passages), whereas in the model with date, we observe a significant increase in $\hat{a}_4$ as well as a decrease in $\hat{a}_5$ over date of composition.

[1] Thanks to an anonymous reviewer for pointing out that area would be expected to vary with length of passage, which prompted me to add this factor.

[2] To perform a backward elimination I began by running a multiple regression on all factors, which included second order interactions date×form, form×composer, date×length, and form×length, all factors with coefficient number, and third order interactions adding coefficient number to all of the second-order interactions just listed. I removed the factor with the highest minimum p-value and no higher-order interactions dependent upon it, until all remaining factors were significant at $p < .01$ or part of some higher-order interaction.

Factor	Coefficient	Significance
Date	-0.0002	—
Expo	0.0242	—
Recap	0.0282	—
$\hat{a}_3$	0.5824	—
$\hat{a}_4$	-5.050	$p < .01$
$\hat{a}_5$	4.766	$p < .01$
Minor	0.0114	—
Date×$\hat{a}_3$	-0.0003	—
Date×$\hat{a}_4$	0.0028	$p < .01$
Date×$\hat{a}_5$	-0.0026	$p < .01$
Expo×$\hat{a}_3$	-0.0008	—
Recap×$\hat{a}_3$	0.0214	—
Expo×$\hat{a}_4$	-0.0346	—
Recap×$\hat{a}_4$	-0.0243	—
Expo×$\hat{a}_5$	0.1080	$p < .0001$
Recap×$\hat{a}_5$	0.0677	$p < .01$
Minor×$\hat{a}_3$	0.0064	—
Minor×$\hat{a}_4$	-0.0212	—
Minor×$\hat{a}_5$	-0.0893	$p < .001$

Table 1. Regression model on distance data. ($R^2 = .852$, Adjusted $R^2 = .827$)

Factor	Coefficient	Significance
Length	0.00002	—
Length×$\hat{a}_3$	-0.00005	—
Length×$\hat{a}_4$	0.00011	—
Length×$\hat{a}_5$	-0.00029	$p < .001$

Table 2. Coefficients for a model replacing date with length in the model for distance data. ($R^2 = .849$, Adjusted $R^2 = .824$)

In the area model, the length factor has more explanatory value, since a longer passage results in more data points, which is likely to increase the area of the convex hulls. In fact, we find a statistically significant increase with length only in $\hat{a}_4$. The other significant effects found in the area model are a) an overall increase with date, b) smaller areas overall for developments, c) smaller areas for Beethoven, and d) larger areas in minor. See table 3 (adjusted $R^2 = .737$, p < .0001).

Figures 5 and 6 show the distance data and trendlines by date for coefficients 4 and 5, separated by formal section. They can be compared to the data for $\hat{a}_3$ in figure 7. We see that $\hat{a}_5$ is much larger overall and decreases over time, while $\hat{a}_4$ increases over time. The lower $\hat{a}_5$ for developments is clear, whereas there is no apparent difference between formal sections in the other coefficients. What appears to be an interaction between formal section and date in $\hat{a}_5$, with recapitulations decreasing much more rapidly over time, did not result in a significant triple interaction (coefficient×date×form) in the regression analysis. Nonetheless, this may reflect that codas, which were included in the recapitulations for the data collection, become longer and more tonally adventurous over the period, especially in Beethoven's works, where they often

Factor	Coefficient	Significance
Date	0.0237	$p < .0001$
Expo	−0.2901	$p < .0001$
Recap	−0.3512	$p < .0001$
$\hat{a}_3$	0.3741	$p < .01$
$\hat{a}_4$	0.1424	—
$\hat{a}_5$	0.0850	—
Haydn	0.2738	$p < .01$
Mozart	0.3041	$p < .01$
Minor	0.1463	$p < .01$
Length	0.0005	—
Length$\times\hat{a}_3$	0.0003	—
Length$\times\hat{a}_4$	0.0017	$p < .001$
Length$\times\hat{a}_5$	0.0002	—

Table 3. Regression model on area data. ($R^2 = .763$, Adjusted $R^2 = .737$)

act as something like a secondary development.

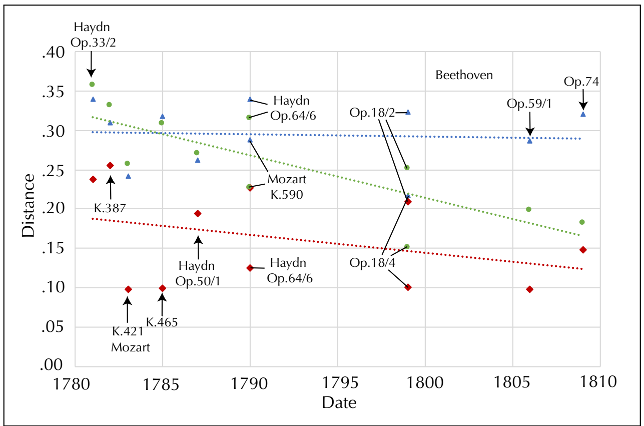

Figure 5. Distance of $\hat{a}_5$ from the origin by date with trendlines for each formal section (blue = expositions, red = developments, green = recapitulations).

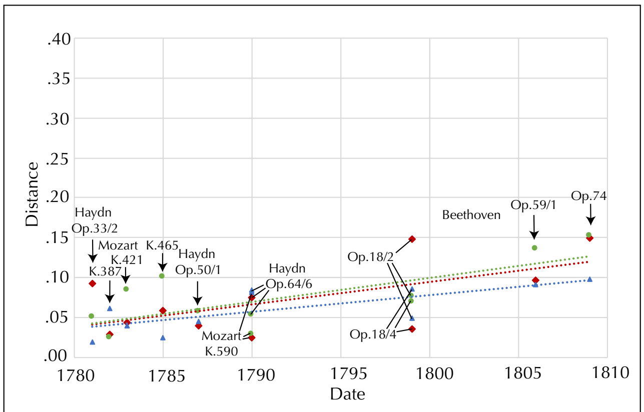

Figure 6. Distance of $\hat{a}_4$ from the origin by date with trendlines for each formal section.

The results for the area data, table 3, mostly consist of simple effects, with the exception of the length to coefficient interaction, which shows that area increases for longer passages only in $\hat{a}_4$. Across all components there is an increase in area by date, evident in figures 8-10, but also lower areas for Beethoven, whose works are all later in date. This means that the composer effect cancels out

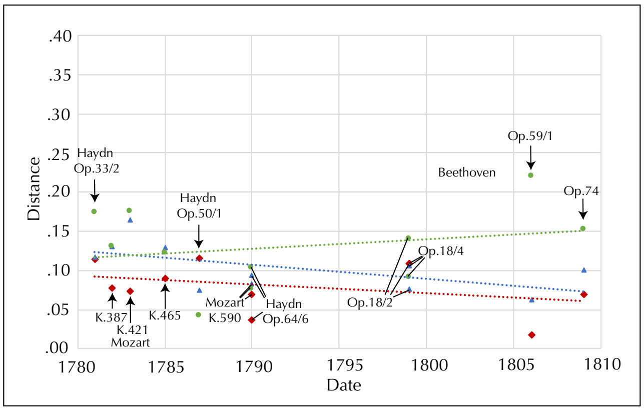

Figure 7. Distance of $\hat{a}_3$ from the origin by date with trendlines for each formal section.

the date effect between composers, so that it only obtains within each composer's oeuvre. That is, the areas increase across each composer's range of dates, but not between composers. Also evident from figures 8-10 is that developments have higher areas across all coefficients, which is the simple effect of form identified by the regression. In addition, the regression found larger areas overall for $\hat{a}_3$ (Fig. 10), and larger areas for minor-key pieces (Mozart K.421 and Beethoven Op. 18 no. 4). The other effect apparent in figures 8-10, the steeper trendline for recapitulations, does show up as a weak effect ($p < .05$) if form$\times$date is added back into the regression model (with a coefficient of 0.0112 for recap$\times$date), but the model of table 3 does not include it because I chose a stronger $\alpha = .01$ criterion in the backward elimination.

4. DISCUSSION

The results confirm and extend much of what we know about diatonicity and enharmonicism, as represented by $\hat{a}_5$, in the late eighteenth and early nineteenth century. While the overall range of activity in $\hat{a}_5$ does not change much from 1780 to 1810 in the repertoire examined, that range shifts closer to the origin, which increases the range of phase values, making enharmonicism, in the form of patterns and relationships that cycle the $\hat{a}_5$ space, more

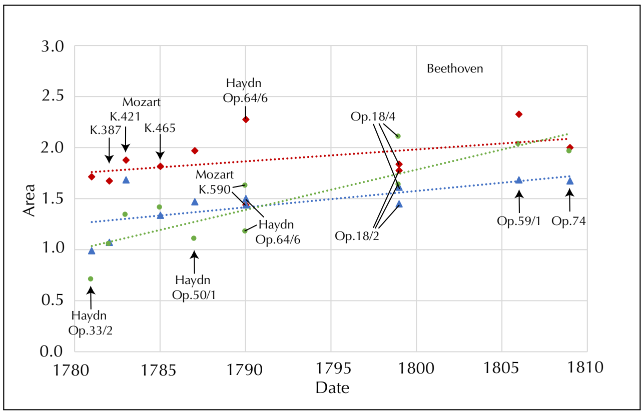

Figure 8. Area of $\hat{a}_5$ by date with trendlines for each formal section (blue = expositions, red = developments, green = recapitulations).

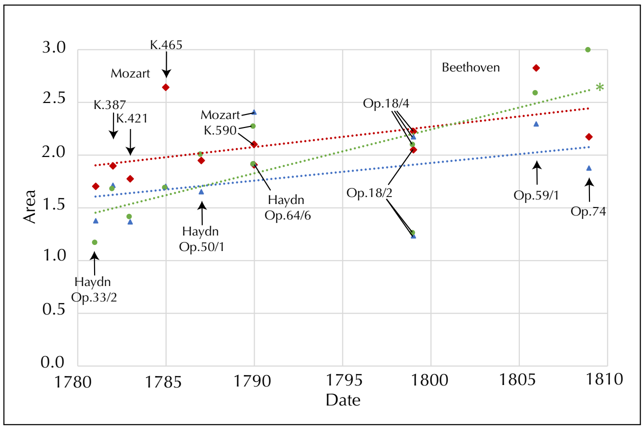

Figure 9. Area of $\hat{a}_4$ by date with trendlines for each formal section.

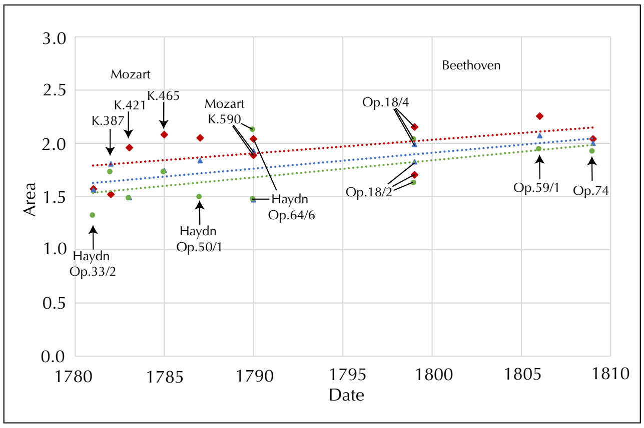

Figure 10. Area of $\hat{a}_3$ by date with trendlines for each formal section.

available. For the other Fourier coefficients, whose ranges of activity have similar or slightly higher areas but considerably smaller distance from the origin, the full range of phase values appear to be consistently available across all of the repertoire examined.

The analysis above did not include data on whether the origin is included within the convex hull, although it was collected for the corpus, because this is largely directly attributable to the area and distance from origin. However, it is a useful criterion in the sense that if the origin is excluded from the convex hull, then the range of phase values does not exceed half of the full cycle. For $\hat{a}_5$, that means that the piece is unlikely to include enharmonic techniques. For other coefficients, it is rare that the origin is excluded from the convex hull, but for $\hat{a}_5$ all analyses with 16- and/or 32-beat windows in the current data pool exclude it except only in five instances. These are development sections of Mozart K.421, Haydn Op. 64/6, Beethoven Op. 18/4, and Op. 59/1 and the recapitulation of Beethoven Op. 74. This list includes some notable examples of enharmonicism for the period. For example, the Mozart K.421 development begins with a sudden shift to E♭ minor, followed by a reinterpretation of E♭ as D♯ in A minor. Haydn Op. 64 no. 6 has a false recapitulation in G♭ major, with the G♭ ultimately reinterpreted as an F♯ in C major. Beethoven's Op. 59/1 has an unusually elaborate development section, which leads to a remarkable climax, where a dramatic E♭ major chord suddenly gives way to a series of enharmonically ambiguous chords. Only the Beethoven example could be considered a true enharmonic cycle, while only the Mozart can be understood as an enharmonic reinterpretation of a chord. In general, though, all of these expand the range of keys enough to make various kinds of enharmonic play possible. The convex hull of $\hat{a}_5$ therefore generalizes across enharmonic techniques, and is promising as a way to computationally detect enharmonicism. It isolates those cases where significant enharmonic distinctions become active, without definitely specifying how the enharmonicism may be compositionally deployed.

A new and interesting result here relating to diatonicity is that as the window size increases (averaging over more music), the range of diatonicity moves further away from zero. This quantifies the musically intuitive idea that diatonicity is a large-scale musical property. Greater musical context typically results in a stronger sense of diatonic position and less diatonic ambiguity. Since diatonicity is essential to the sense of key, this is consistent with the idea that keys typically govern long stretches of music. This is not true of other coefficients, whose ranges of activity stay relatively close to zero regardless of window size.

The most surprising results here have to do not with diatonicity but with $\hat{a}_4$. This coefficient has been associated with octatonic harmony in twentieth-century music [2, 20, 21] and seventh chords in tonal music [19]. It may also relate to the use of diminished seventh chords and describe a Riemannian concept of function (along the lines, e.g., of De Jong and Noll [10]). This dimension of harmony appears to play a greater role over the historical time period examined. Its range of activity shifts away from the center of the space, indicating that a particular phase value of $\hat{a}_4$ begins to act as a reference point (much as the central $\hat{a}_5$ phase value represents the basic scale of the home key). This is particularly notable in that it opposes the trend on all other coefficients (especially $\hat{a}_5$) to decrease over time[3]. At the same time, however, the *area* of $\hat{a}_4$ gets larger, so that range of phase values remains consistent. Taken together, we may say then that larger $|\hat{a}_4|$ values, caused probably by heavier use of diminished seventh chords, become more prevalent, and the association of particular diminished sevenths (enharmonically) with particular functions becomes more established. While octatonicism has often been recognized as an important aspect of the break with tonality on the part of early twentieth-century composers like Stravinsky, Ravel, Debussy, Bartòk and Scriabin [4, 7, 8, 12, 14, 15, 18, 20, 21], and Liszt is usually the earliest composer credited with using an octatonic scale [17], the results here hint at the possibility that the seeds of octatonicism are planted much deeper in the history of tonality.

[3]Given that we are talking about power-normalized Fourier coefficients, and that the Fourier transform conserves total power (Parseval's theorem), an overall trend of decreasing coefficient values necessarily indicates an increase in $\hat{a}_0$, corresponding to general chromaticism, or the tendency to use all pitch-classes relatively equally.

5. ACKNOWLEDGMENTS

Many thanks to Harrison Davis, who wrote the python code used for the data collection in this study.

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Texte édité par Corentin Guichaoua.